Comparison of wavefront sensing using subdivision at the aperture and focal planes

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Abstract
The atmosphere introduces a phase distortion on the incoming wavefront from an astronomical object, causing a speckle image at the ground-based telescope. Wavefront sensing is a set of methods to estimate this phase distortion which can then be used to improve the captured image. The Shack-Hartmann wavefront sensor consists of a lenslet array placed in the aperture plane of the telescope which subdivides the complex field in the aperture plane. The lenslet array can also be used to achieve subdivision at the focal plane. We present a Fourier analysis of the latter approach and compare with the Shack-Hartmann sensor.

Keywords: adaptive optics, wavefront sensing, Shack-Hartmann sensor.

1 Introduction
Ground-based telescopes have long been constrained by the effect of the atmosphere which has degraded the quality of the images formed. The effect of the atmosphere is to produce a random time-varying phase aberration on an incoming wavefront resulting in a speckle image at the ground-based telescope as shown in Fig. 1(a). The atmosphere can degrade the resolution of the images by a factor of 20 or more.

Recently, adaptive optics [1] has proved that the limits posed by the atmosphere can be overcome, with diffraction-limited resolution being demonstrated on large telescopes. Conventional closed loop adaptive optics, Fig. 2(a), uses a wavefront sensor to detect the atmospheric distortion which feeds back to a deformable mirror to compensate for this distortion. Alternatively, computer post-processing, Fig. 2(b), still uses a wavefront sensor to estimate the distortion but has no feedback loop. This eliminates the need for a deformable mirror, and enables a much cheaper system to be developed. Deconvolution from wavefront sensing uses this approach [2] by combining a set of short exposure images, such as Fig. 1(b), to produce a turbulence compensated image of the object, Fig. 1(c).

The most commonly used wavefront sensor in astronomical imaging is the Shack-Hartmann wavefront sensor shown in Fig. 3. The Shack-Hartmann wavefront sensor consists of an array of lenslets placed in the aperture plane of the telescope. The lenslet array subdivides the complex field in the aperture with each lenslet forming a low resolution image of the object. When there is no aberration present these images are focused onto points directly below the centre of the respective lenslet as in Fig 3(a). However, if there is an overall mean wavefront slope over the lenslet then that image is displaced from the centre by an amount proportional to the mean wavefront slope [2], Fig 3(b). The entire wavefront can then be reconstructed from the mean slope measurements across the aperture.

Fig. 4(c) shows an alternative method for wavefront sensing derived from positioning the lenslet array at the focal plane of the telescope. Geometric optics predicts that light travels in a direction perpendicular to the wavefront. If there is no distortion then all light would arrive at a single point in the focal (or image) plane as shown in Fig. 4(a). A slope at any point in the aperture causes the light from this point to be deflected in the focal plane as shown in Fig. 4(b). When the focal plane is subdivided by a lenslet array as shown in Fig. 4(c) light from points in the aperture where the wavefront is undistorted (slope zero) passes through the central lenslet. Light from points in the aperture where the slope is more positive than expected pass through the lenslet on the right while points where the slope is more negative than expected pass through the lenslet on the left. The final step in this wavefront sensor configuration is that each lenslet is designed to form an image of the aperture. Thus in Fig. 4(c), if a pixel in the image on the right is illuminated, this implies that the light passed through the lenslet on the right and hence the slope at this point in the aperture is positive. Conversely, if a pixel on the left image is illuminated this implies that the slope at this point in the aperture is negative. Thus the lenslet array at the focal plane is equivalent to a series of bandpass filters on the slope at each point in the aperture. The special case of a \(2\times2\) array of lenslets is also known as the pyramid sensor[3].

This paper compares the wavefront sensing capabilities for the use in astronomical imaging of the lenslet array at the aperture and focal planes. These two wavefront
sensing schemes are linked by a Fourier transform relationship since the complex field at the aperture and focal planes form a Fourier transform pair. Section 2 discusses the Shack-Hartmann wavefront sensor in more detail. The basis for wavefront sensing at the focal plane with a lenslet array is made in Section 3. A discussion of the duality between the two sensors is made in Section 4.

2 Subdivision at the aperture plane

The Shack-Hartmann wavefront sensor subdivides the complex field in the aperture plane with a lenslet array. Each lenslet forms a low resolution image from the field over it. When there is no aberration, Fig. 5(a), the images are formed directly below the lenslet. When the wavefront is aberrated with atmospheric turbulence the images are displaced by a distance proportional to the mean wavefront slope over the lenslet, Fig. 5(b).

The wavefront slope in the aperture plane in the $x$ and $y$ directions over each lenslet can be formed by calculating the displacement of the image in the $x$ and $y$ directions from the unaberrated rest positions. The displacement is conventionally estimated with the centroid estimator which computes the centre of mass of the image. If the detector consists of an array of finite sized pixels of width $\Delta$ with $(2P, 2Q)$ pixels per image the centroid estimator in the $x$ and $y$ directions is

$$
\hat{s}_x = \frac{\sum_{p=-P}^{P} \sum_{q=-Q}^{Q} I(i\Delta, j\Delta)(p\Delta - \delta_x)}{\sum_{p=-P}^{P} \sum_{q=-Q}^{Q} I(i\Delta, j\Delta)}
$$

$$
\hat{s}_y = \frac{\sum_{p=-P}^{P} \sum_{q=-Q}^{Q} I(i\Delta, j\Delta)(q\Delta - \delta_y)}{\sum_{p=-P}^{P} \sum_{q=-Q}^{Q} I(i\Delta, j\Delta)}
$$

where $(\delta_x, \delta_y)$ is the offset from the first pixel to the origin in the focal plane. The wavefront is reconstructed from the centroids as a sum of basis polynomials. For a circular aperture a typical set of basis polynomials are the Zernikes [4]. The Zernike weights, $a$, are reconstructed via the equation [5]:

$$
a = (k_{zz}^{-1} + \Theta^T K_{nn}^{-1} \Theta)^{-1} \Theta^T k_{nn}^{-1} s
$$

where $s$ is a vector containing the centroid measurements. The matrix $\Theta$ is called the interaction matrix.
matrix \([6]\) because it maps the slope measurements to the weights of the bases. \(K_{zz}\) is the covariance matrix for the Zernike polynomial coefficients, while \(K_{nn}\) is the noise covariance matrix.

The size of the lenslet determines the spatial resolution of the Shack-Hartmann sensor. When the resolution is low, only a small number of modes in the atmospheric turbulence can be determined by the sensor. The smaller the lenslet the smaller a region of the wavefront the slope can be estimated for. The aberrations of a higher order than tilt over each lenslet cannot be detected. However, reducing the size of the lenslets in the array reduces the number of photons per lenslet. This results in a reduction in the accuracy with which the slope estimates can be made. Also if the lenslet size is made smaller than the Fried parameter, \(r_0\) \([7]\), then diffraction effects mean the spot size increases and consequently the accuracy of the slope estimate decreases.

In practice the optimum for the Shack-Hartmann sensor is when the lenslet size is close to \(r_0\).

### 3 Subdivision at the focal plane

The lenslet array subdivides the complex field in the focal plane and each lenslet forms an image of the aperture in the conjugate aperture plane. The images in the conjugate aperture plane created with a lenslet array at the focal plane are shown in Fig. 6(a) and (b) for no atmospheric turbulence and with atmospheric turbulence respectively. When there is no atmospheric turbulence the central four lenslets are illuminated equally.

#### 3.1 Mathematical analysis of the lenslet array at focal plane

Mathematically, the presence of a lens in the focal plane performs the inverse Fourier transform...
relationship to produce an image in the conjugate aperture plane, described by co-ordinates \((\xi, \eta)\). The complex field in the conjugate plane is a low-pass filtered image of the aperture. The filtering is due to the finite size of the lens, which we describe mathematically as a spatial filter \(H(\xi, \eta)\). The resulting image in the conjugate aperture plane, \(I(\xi, \eta)\), is given by, [8]

\[
I(\xi, \eta) \propto \left| \mathcal{F}^{-1} \left[ H(u, v) \mathcal{F} \left[ P(\xi, \eta) \exp[j\phi(\xi, \eta)] \right] \right] \right|^2.
\]  

Equation (3) can be expanded by making use of the convolution theorem to produce

\[
I(\xi, \eta) \propto \left| \mathcal{F}^{-1} \left[ \mathcal{F}^{-1} [H(u, v)] \otimes \mathcal{F}^{-1} [P(\xi, \eta) \exp[j\phi(\xi, \eta)]] \right] \right|^2.
\]  

Using the linearity of the Fourier transform and the convolution operator Eq. (4) simplifies to

\[
I(\xi, \eta) \propto \left| h(\xi, \eta) \otimes P(\xi, \eta) \exp[j\phi(\xi, \eta)] \right|^2.
\]  

Since a lenslet traditionally consists of square lenses we assume a square lenslet with a linear dimension of \(d\) centred at a point \((u', v')\) in the focal plane. The corresponding spatial filter, \(H(u, v)\), for this lenslet is

\[
H(u, v) = \begin{cases} 
1 & u' - \frac{d}{2} \leq u \leq u' + \frac{d}{2} \\
1 & v' - \frac{d}{2} \leq v \leq v' + \frac{d}{2} \\
0 & \text{otherwise}
\end{cases}
\]  

(6)

and the IFT of this spatial filter, \(h(\xi, \eta)\), can be calculated as:

\[
h(\xi, \eta) = d^2 \frac{\sin(\pi \xi d)}{\pi \xi d} \exp[j2\pi \xi u'] \frac{\sin(\pi \eta d)}{\pi \eta d} \exp[j2\pi \eta v'].
\]  

(7)

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Figure 5: The images produced from a 8×8 array of lenslets at the aperture plane with a circular aperture when (a) there is no atmospheric turbulence, and (b) when there is atmospheric turbulence. The aberrated phase screen is shown in (c).

(a)  
(b)  
(c)

Figure 6: The images resulting from a 4×4 lenslet array placed at the focal plane with a circular aperture when (a) there is no atmospheric turbulence, and (b) when there is atmospheric turbulence. The aberrated phase screen is shown in (c).
The image formed from this lenslet in the focal plane is given by substituting Eq. (7) into Eq. (5).

\[ I(\xi, \eta) \propto |d^2 \text{sinc}(\pi \xi d) \exp[j2\pi \xi u'] \text{sinc}(\pi \eta d) \exp[j2\pi \eta v'] \otimes P(\xi, \eta) \exp[j\phi(\xi, \eta)]|^2. \quad (8) \]

The image formed from a lenslet in the focal plane is the magnitude squared of the convolution of a two-dimensional sinc function with the complex amplitude in the aperture. The effect of convolving with the sinc function is to smear the image \( I(\xi, \eta) \) and limit the resolution with which the slopes can be determined in the aperture. The lobe width of the sinc function is determined by the width of the lenslet in the focal plane and the phase by the position of the lenslet in the focal plane. Because of the Fourier relationship between the focal and conjugate aperture planes, as the size of the lenslets increases then the width of the main lobe of the sinc function of Eq. (8) decreases. Thus as the lenslet size increases the spatial resolution improves.

### 3.2 Slope filtering

It is the linear phase term of the two-dimensional sinc function in Eq. (8) that isolates the slopes in the aperture. Assuming the scintillation in the aperture is small and that the phase can be expressed as a pure tilt in the \( \xi \) direction only, the complex field in the aperture, \( P(\xi, \eta) \exp[j\phi(\xi, \eta)] \), simplifies to \( \exp[j2\pi k \xi] \), where \( k \) is the coefficient of the tilt aberration. Eq. (8) simplifies to

\[ I(\xi, \eta) \propto |d^2 \text{sinc}(\pi \xi d) \exp[j2\pi \xi u'] \text{sinc}(\pi \eta d) \exp[j2\pi \eta v'] \otimes \exp[j2\pi k \xi]|^2. \quad (9) \]

In order to simplify the analysis of the problem we assume that the telescope has a square aperture of dimension \( D \). Expansion of Eq. (9) with Euler’s identity and application of the definition of the convolution integral yields

\[ I(\xi, \eta) \propto \left| \int_{-\frac{D}{2}+\eta}^{\frac{D}{2}+\eta} \int_{-\frac{D}{2}+\xi}^{\frac{D}{2}+\xi} \exp[j2\pi k (\xi - \xi')] \right|^2 \frac{\exp[j\pi(\xi')(d + 2u')] - \exp[-j\pi(\xi')(d - 2u')]}{2j\pi \xi'} \frac{\exp[j\pi(\eta')(d + 2v')] - \exp[-j\pi(\eta')(d - 2v')]}{2j\pi \eta'} \, d\xi' \, d\eta'. \quad (10) \]

where \( \xi' \) and \( \eta' \) are the dummy integration variables. Computing this integral over \( \xi' \) and \( \eta' \) results in

\[ I(\xi, \eta) \propto \left| \frac{1}{4\pi^2} \left( E_i[-\frac{j\pi}{2}(2\eta - D)(d - 2v')] - E_i[-\frac{j\pi}{2}(2\eta + D)(d - 2v')] + E_i[-\frac{j\pi}{2}(2\eta - D)(d + 2v')] - E_i[-\frac{j\pi}{2}(2\eta + D)(d + 2v')] \right) \right|^2 \]

where \( E_i(x) \) is the Exponential Integral Function defined by

\[ E_i(x) = \int_x^\infty \frac{\exp[t]}{t} \, dt. \quad (12) \]

The plot of intensity versus the wavefront tilt, \( k \), from Eq. (11) shows how the lenslet acts as a passband filter on the slopes. The centre of the passband is equal to the lenslet centre, \( d' \), and the passband width equal to the width of the lenslet, \( d \). When the slope of the wavefront at a given point \( (\xi, \eta) \) lies in the passband of the slope filter defined by the lenslet then the intensity at the same point in the re-imaged aperture is approximately one. If the slope at this given point \( (\xi, \eta) \) lies outside the passband of the slope filter of the lenslet then the intensity at the same point in the re-imaged aperture is approximately zero. There is, however, also ringing in the passband of the filter and at the edges of the stopband as can be seen in Fig. 7(a),(b) and (c). The width of the transition region between the pass and stop bands of the slope filter is determined by the width of the diffraction-limited spot.

The lenslet at the focal plane can therefore tell us the range that the slope at a point in the aperture lies within. As the lenslet size decreases, and consequently the width of the passband of the slope filter decreases (Fig. 7(a) cf. 7(b)), we can determine the slope to a greater accuracy.

### 3.3 Reconstruction from lenslet array in the focal plane

The wavefront slope estimates in the orthogonal \( \xi \) and \( \eta \) directions can be estimated from a weighted sum of the images. The weighting is determined by the distance from the centre of the lenslet, from which the image came, to the centre of the lenslet array, and hence origin in the focal plane.

The slope estimates in the \( \xi \) and \( \eta \) directions for an array of \( 2M \times 2N \) array of lenslets are given by
if the lenslet size is less than the Shack-Hartmann sensor. Firstly, in the Shack-Hartmann sensor behavior in partial correction adaptive optics systems,” A. & A.

4 Conclusion

The residual error in wavefront sensing is a function of the spatial resolution of the slope estimates and the accuracy with which these slope estimates are made. It is well established that for the Shack-Hartmann sensor there is a trade-off between these two quantities and that the trade-off is determined by the size of the lenslets at the aperture plane. As the lenslet size increases the spatial resolution gets worse and the slope accuracy improves and the slope accuracy decreases.

Subdivision with a lenslet array at the focal plane has the potential to perform better than the Shack-Hartmann sensor. Firstly, in the Shack-Hartmann sensor diffraction effects cause the spot size to increase if the lenslet size is less than \( r_0 \) meaning that the size of the lenslets at the aperture plane is effectively limited. This is not the case with the lenslet array at the focal plane. Secondly, the Shack-Hartmann sensor is blind to modes which are even symmetric over each lenslet.

The lenslet array at the focal plane can reconstruct these modes.

The duality between the formation of the slope estimates from the images when subdividing in the aperture and focal planes should also be noted. For the Shack-Hartmann sensor the slope over a region (lenslet) is given by the centroid of the corresponding image. For the lenslet array at the focal plane the slope estimate for each region (pixel) is given as a centroid of the aperture images for the corresponding region (pixel). Comparing the two sets of slope estimation formulae, Eq. (1) and Eq. (13), it can be seen that increasing the number of lenslets \((M,N)\) in the array in the focal plane is analogous to increasing the number of pixels \((P,Q)\) used to detect each image in the Shack-Hartmann sensor.

The Zernike weights, \(a\), are again reconstructed via

\[
a = (K_{nm}^{-1} + \Theta^T K_{mm}^{-1} \Theta)^{-1} \Theta^T K_{mm}^{-1} s
\]

except here \(\Theta\) is formed from the partial derivatives of the Zernike modes as set out by Noll [4] and \(s\) is a vector containing the slopes in the \(\xi\) and \(\eta\) directions.

References